QUOTIENT POLYTOPES OF CYCLIC POLYTOPES PART I: STRUCTURE AND CHARACTERIZATION†

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ABSTRACT

We investigate the quotient polytopes C/F, where C is a cyclic polytope and F is a face of C. We describe the combinatorial structure of such quotients, and show that under suitable restrictions the pair (C, F) is determined by the combinatorial type of C/F. We describe alternative constructions of these quotients by "splitting vertices" of lower-dimensional cyclic polytopes. Using Gale diagrams, we show that every simplicial d-polytope with $d+3$ vertices is isomorphic to a quotient of a cyclic polytope.

Introduction

The cyclic polytopes, which were discovered early this century by Carathedory [3, 4] and more recently rediscovered by Gale [5] and Motzkin [12], play an important role in the combinatorial theory of convex polytopes. The main reason for this is the fact that they form the simplest case of simplicial neighborly polytopes, which, as proved by McMullen [10], have the largest number of faces of each dimension, among all the polytopes of the same dimension and with the same number of vertices. For a short history and further information about cyclic polytopes the reader should consult [6].

In a series of papers, the first of which is presented here, we intend to investigate a certain family of polytopes that is derived from the cyclic polytopes, namely, the quotient polytopes of cyclic polytopes. For every polytope P and for every face F of P, the *quotient* P/F is a polytope whose face-lattice is isomorphic

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to the sublattice $\{G : F \subset G \subset P\}$ of the face-lattice of P. When F is a vertex of P, the quotient P/F is simply the vertex figure of P at F. The notation P/F for quotients was first introduced by McMullen and Shephard in [11, page 71]. The concept itself appears already in [6, exercise 3.4.10 (iii)].

Quotient polytopes of cyclic polytopes were investigated also by Hering [9] and by Schönwald [14]. The approach and purpose of Hering and Schönwald are completely different from ours, and few of their results overlap with ours. We feel that a brief description of the different approaches would be appropriate at this point.

The combinatorial structure of an $(n-r)$-dimensional cyclic polytope C with n vertices is shown by Hering [9] to be reflected in the alternating binary sequence $d = (0, 1, 0, \cdots)$ of length n together with the set of all the alternating subsequences of d of length r, and a certain partial order relation defined on this set. The structure of a quotient of C is similarly reflected in a sequence d_1 that is obtained from d by *omitting* certain digits. Moreover, every such binary sequence corresponds to a quotient of a cyclic polytope (Hering uses the term "binary polytope" for what we call a quotient polytope of a cyclic polytope). This is proved by Hering by means of his general "simplex diagram" method. Using this approach Hering proves the Upper Bound Theorem for quotient polytopes of cyclic polytopes, and a certain generalization of the inequality of the arithmetic and geometric means.

Schönwald [14] follows Hering's approach, and his main interest is to develop a more direct geometric realization of the binary sequence d_1 as a convex polytope. For this purpose he considers the sequence d_1 as obtained from a smaller alternating binary sequence $d_2 = (0, 1, 0, 1, \cdots)$ by *addition* of several 0 and 1 digits. (d_2 is obtained from d_1 by replacing each block of 0's (resp. 1's) by a single 0 (resp. 1)). A realization of this structure as a convex polytope, given by Schönwald, is obtained by considering a cyclic polytope C_2 that realizes the sequence d_2, replacing each vertex of C_2 by a suitable simplex, and taking the convex hull. In Section 6 we deal with Schönwald's construction from our point of view.

The main topics of our investigation are: the reconstruction of C and F from C/F, where C is a cyclic polytope and F a face of C; characterizations of quotients of cyclic polytopes; the combinatorial automorphisms of C/F; enumeration of combinatorial types of quotients C/F; the degree of neighborliness of C/F; the f-vector of C/F, and the effect of "slight" changes in F on the f-vector of C/F.

The first three sections of this paper serve as an introduction to the entire work. Here we define the basic terms and concepts that will be used later. In Section 1 we define quotients of convex polytopes and simplicial complexes. In Section 2 we define *missing faces*, and establish some of their properties. The notion of a missing face turns out to be a powerful tool for investigating simplicial polytopes and more general simplicial complexes. We feel that the usefulness of this concept goes beyond the realm of this work.

In Section 3 we describe the cyclic polytopes and their quotients, which form the topic of the present investigation. We show that every quotient of a cyclic polytope is combinatorially isomorphic to a quotient C/F, where C is an even-dimensional cyclic polytope, and F is a face of C, such that the set vert F is "separated" on the moment curve by the remaining vertices of C. This enables us to restrict our attention to such quotients C/F, where C is even-dimensional and F is "separated". The advantage of C being even-dimensional is that we can regard the vertices of C as ordered *cyclically* on the moment curve.

In Sections 4 and 5 we investigate in detail the quotients C/F, where $C = C(v, 2m)$ is a cyclic $2m$-polytope with v vertices, and F is a "separated" face of C, $|\text{vert}\, F| = j$. In Section 4 we deal mainly with the case where $v \geqq 2m + 3$ and $j < m$. We show that in this case C and F can be essentially reconstructed from C/F, and we also determine the combinatorial automorphisms of C/F.

In Section 5 we deal with the remaining cases, where $v \leqq 2m + 2$ or $j = m$ ($j > m$ is impossible). It turns out that in these cases C/F is (isomorphic to) a direct sum of one or more simplices.

In Section 6, which concludes the first part of this work, we deal with Schönwald's construction mentioned above from our point of view; we also describe it in terms of Gale-diagrams. One of the few results common to our work and to [9] is Theorem 6.9, which says that every simplicial d polytope with $d + 3$ vertices is combinatorially isomorphic to a quotient of a cyclic polytope.

A brief remark about the origins of this work is appropriate. In 1971 the first author investigated the vertex figures of cyclic polytopes (see [1]). The second author solved the questions left open by the first author, and suggested dealing with the more general topic of quotient polytopes of cyclic polytopes. The combined effort started when we met in August 1972, and led to the series of papers of which the first is presented here, and which includes the results about vertex figures obtained in 1971 by the first author while at Temple University, Philadelphia, Pa.

Our terminology and notation follows [6]. To denote the end of a proof we use the sign □.

1. Quotients of simplicial complexes and convex polytopes

Let K be a convex polytope, and p a vertex of K. The *vertex figure* K_p of K at p is defined [6, p. 49, ex. 8] as the intersection of K with a hyperplane H which strictly separates the vertex p from the remaining vertices of K. The face-lattice $\mathcal{F}(K_p)$ of K_p is isomorphic to the upper segment of $\mathcal{F}(K)$ determined by $\{p\}$. If $\Psi \in \mathcal{F}(K)$, $p \in \Psi$, and Ψ' $(= \Psi \cap H)$ is the corresponding face of K_p, then $\dim \Psi' = \dim \Psi - 1$.

Repeated application of the above construction establishes the following:

If $K \subset R^d$ is a convex polytope, and Φ a k-face of K, then there exists a polytope K_Φ whose face-lattice is isomorphic to the upper segment of $\mathcal{F}(K)$ determined by Φ. If $\Psi \in \mathcal{F}(K)$, $\Phi \subset \Psi$, then the corresponding face Ψ' of K_Φ satisfies $\dim \Psi' = \dim \Psi - \dim \Phi - 1$. K_Φ can be realized as the intersection of K with a suitable $(d-k-1)$-flat. K_Φ, and any polytope combinatorially equivalent to K_Φ, is called a *quotient polytope* (or simply a *quotient*) of K determined by Φ, and is denoted by K/Φ. From the definition it follows that a quotient of a quotient of K is again a quotient of K. An alternative construction of quotient polytopes, using duality, may be found in [11, pp. 71–72], and in [6, exercise 3.4.10 (iii)].

If K is a simplicial polytope, then the boundary complex $\mathcal{B}(K)$ $(= \mathcal{F}(K)\setminus\{K\})$ is a simplicial complex. Since we shall be dealing only with combinatorial properties of simplicial polytopes, we regard each face as its set of vertices. Under this point of view, $\mathcal{B}(K)$ becomes an abstract simplicial complex, i.e., a finite collection of sets, which contains together with any element S all the subsets of S.

If $\mathcal{D}$ is an abstract simplicial complex, we shall refer to its elements (cells) as "faces of $\mathcal{D}$". The *facets* of $\mathcal{D}$ are its maximal faces (maximal with respect to inclusion). If F is a face of $\mathcal{D}$, let $\dim F = |F| - 1$, and let $\dim \mathcal{D} = \max\{\dim F : F \in \mathcal{D}\}$. We call F a d-face, or $\mathcal{D}$ a d-complex, if $\dim F = d$, or $\dim \mathcal{D} = d$, respectively. $\mathcal{D}$ is a *pure* (homogeneous) d-complex if all its facets are d-faces. Thus the boundary complex of a simplicial d-polytope is a pure $(d-1)$-complex. If J is a face of $\mathcal{D}$, define the abstract quotient complex $\mathcal{D}/J$ by

$$(*) \qquad \mathcal{D}/J = \{S \setminus J : S \in \mathcal{D}, J \subset S\} = \{T : T \cap J = \emptyset, T \cup J \in \mathcal{D}\}.$$

$\mathcal{D}/J$ is again an abstract simplicial complex. (If $J \notin \mathcal{D}$, then (*) yields $\mathcal{D}/J = \emptyset$. We shall occasionally use the "quotient" $\mathcal{D}/J$, as defined by (*), where $\mathcal{D}$ is an arbitrary collection of finite sets, and J is any finite set.)

$\mathcal{D}/J$ is, in fact, the link of J in $\mathcal{D}$ (see [6, page 40]). The complex $\mathcal{D}/J$ is naturally isomorphic to the upper segment of $\mathcal{D}$ determined by J, under the correspondence $T \to T \cup J$. Thus, if K is a simplicial polytope and $\Phi \in \mathcal{B}(K)$, then $\mathcal{B}(K/\Phi)$ is naturally isomorphic to $\mathcal{B}(K)/\Phi$. Therefore, instead of dealing with the "geometric" quotients K/Φ, we shall study the "abstract" quotients $\mathcal{B}(K)/\Phi$.

2. Missing faces

Missing faces, defined below, play a central role in this paper. They seem to be a potentially useful tool in the study of simplicial polytopes and complexes in general.

DEFINITION 2.1. Let $\mathcal{D}$ be an abstract simplicial complex. A set M of vertices of $\mathcal{D}$ is a *missing face* (of $\mathcal{D}$) if $M \notin \mathcal{D}$, but every proper subset of M belongs to $\mathcal{D}$. We denote the set of missing faces of $\mathcal{D}$ by $\operatorname{mf} \mathcal{D}$.

If follows from the definition that every missing face of $\mathcal{D}$ contains at least two vertices (unless $\mathcal{D} = \emptyset$), that a missing face cannot properly include another missing face, and that $\operatorname{mf} \mathcal{D} \neq \emptyset$, unless $\mathcal{D}$ contains all subsets of $\operatorname{vert} \mathcal{D}$, i.e., unless $\operatorname{vert} \mathcal{D} \in \mathcal{D}$.

If K is a simplicial polytope, then we write $\operatorname{mf} K$ for $\operatorname{mf} \mathcal{B}(K)$, and call the elements of $\operatorname{mf} K$ missing faces of K. (Note that under this convention, if K is a d-simplex, $d \geqq 1$, then $\operatorname{vert} K \in \operatorname{mf} K$.)

The most important property of $\operatorname{mf} \mathcal{D}$ is that $\mathcal{D}$ can be reconstructed from $\operatorname{vert} \mathcal{D}$ and $\operatorname{mf} \mathcal{D}$, as follows:

LEMMA 2.2. *If $\mathcal{D}$ is an abstract simplicial complex and $S \subset \operatorname{vert} \mathcal{D}$, then $S \in \mathcal{D}$ iff no subset of S belongs to* $\operatorname{mf} \mathcal{D}$.

PROOF. If $S \in \mathcal{D}$ and $T \subset S$, then $T \in \mathcal{D}$, and therefore $T \notin \operatorname{mf} \mathcal{D}$. Conversely, if $S \subset \operatorname{vert} \mathcal{D}$ but $S \notin \mathcal{D}$, let M be a minimal subset of S which is not in $\mathcal{D}$. Then $M \in \operatorname{mf} \mathcal{D}$. □

The boundary complex $\mathcal{B}(K)$ of a simplicial polytope K can be reconstructed from $\operatorname{mf} K$ alone, as follows:

LEMMA 2.3. *If K is a simplicial polytope*, $\dim K \geqq 1$, *then every vertex of K belongs to a missing face of K.*

PROOF. If $p \in \operatorname{vert} K$, let F be a facet of K which does not contain p. Then $\{p\} \cup F \notin \mathcal{B}(K)$. Let T be a minimal subset of F such that $\{p\} \cup T \notin \mathcal{B}(K)$. Then $\{p\} \cup T \in \operatorname{mf} K$. □

THEOREM 2.4. *If K is a simplicial polytope, then $\mathcal{B}(K)$ can be reconstructed from* $\operatorname{mf} K$.

PROOF. If $K = \varnothing$, then $\mathcal{B}(K) = \varnothing$, $\operatorname{mf} K = \{\varnothing\}$. If $\dim K = 0$, then $\mathcal{B}(K) = \{\varnothing\}$, $\operatorname{mf} K = \varnothing$. If $\dim K \geqq 1$, then $\operatorname{vert} K = \bigcup \operatorname{mf} K$, by Lemma 2.3, and $\mathcal{B}(K)$ can be reconstructed by Lemma 2.2. □

For an abstract simplicial complex $\mathcal{D}$, denote by $\max \mathcal{D}$ the set of facets of $\mathcal{D}$. Note that $\mathcal{D}$ is the set of all subsets of elements of $\max \mathcal{D}$, and $\operatorname{vert} \mathcal{D} = \bigcup \mathcal{D} = \bigcup \max \mathcal{D}$. It follows that if $\mathcal{D}$, $\mathcal{D}'$ are abstract simplicial complexes, $\operatorname{vert} \mathcal{D} = V$, $\operatorname{vert} \mathcal{D}' = V'$, and $\varphi : V \to V'$ is a bijection, then φ induces an isomorphism between $\mathcal{D}$ and $\mathcal{D}'$ iff φ induces an isomorphism between $\max \mathcal{D}$ and $\max \mathcal{D}'$, or between $\operatorname{mf} \mathcal{D}$ and $\operatorname{mf} \mathcal{D}'$ (i.e., $\mathcal{D}' = \{\varphi(S) : S \in \mathcal{D}\} \leftrightarrow \max \mathcal{D}' = \{\varphi(F) : F \in \max \mathcal{D}\} \leftrightarrow \operatorname{mf} \mathcal{D}' = \{\varphi(M) : M \in \operatorname{mf} \mathcal{D}\}$).

Also note that if J is a face of $\mathcal{D}$, then $\max(\mathcal{D}/J) = (\max \mathcal{D})/J$. This identity will be used in the sequel.

3. Cyclic polytopes

The moment curve M_d in R^d ($d \geqq 2$) is defined parametrically by

$$x(\tau) = (\tau, \tau^2, \cdots, \tau^d) \qquad (-\infty < \tau < \infty).$$

A *cyclic d-polytope* with v vertices $C(v,d)$ ($v > d$) is the convex hull of v distinct points on M_d (or any polytope combinatorially isomorphic to it). $C(v,d)$ is a simplicial $[d/2]$-neighborly polytope, i.e., every $[d/2]$ vertices of $C(v,d)$ determine a face. For a detailed treatment of cyclic polytopes see [6, Section 4.7] and [11, pp. 82–90].

Now suppose $C(v,d) = \operatorname{conv}\{x_i : 1 \leqq i \leqq v\}$, where $x_i = x(\tau_i) = (\tau_i, \tau_i^2, \cdots, \tau_i^d)$ and $\tau_1 < \tau_2 < \cdots < \tau_v$. The combinatorial structure of $C(v,d)$ is determined by the following rule, known as "Gale's evenness condition":

Let $V = \{1, \cdots, v\}$. If $S \subset V$, then $\{x_i : i \in S\}$ is a facet of $C(v,d)$ iff $|S| = d$, and

(*) between any two members of $V \setminus S$
there is an even number of members of S.

Now, if $S \subset V$ and $v \in S$, then condition (*) for V and S is clearly equivalent to the corresponding condition for $S \setminus \{v\}$ and $V \setminus \{v\}$, i.e., between any two members of $(V \setminus \{v\}) \setminus (S \setminus \{v\})$ there is an even number of members of $S \setminus \{v\}$. Therefore, if $d \geqq 3$, then $\max \mathcal{B}(C(v,d))/\{x_d\}$ is naturally isomorphic to

$\max \mathscr{B}(C(v-1,d-1)$, or, in other words, the vertex figure of $C(v,d)$ at x_v is of type $C(v-1,d-1)$. The same holds for the vertex figure of $C(v,d)$ at x_1.

Therefore, every quotient of an odd dimensional cyclic polytope $C(v-1,2m-1)$ is also a quotient of an even dimensional cyclic polytope $C(v,2m)$. So from now on we shall restrict our attention to the case where $d=2m$ is even.

In order to facilitate the combinatorial manipulation of the boundary complex of $C(v,2m)$, we introduce an abstract complex $\mathscr{C}(v,2m)$ as follows:

Let $V=\{i_1,\cdots,i_v\}$ be a finite set of integers, $i_1<i_2<\cdots<i_v$, $v\geqq 3$. The natural ordering of V induces a cyclic structure $C(V)$, as follows: $C(V)$ is an undirected graph with vertex set V. The edges of $C(V)$ are the pairs $\{i_\nu,i_{\nu+1}\}$ $(1\leqq \nu<v)$ and $\{i_v,i_1\}$.

Let S be a nonempty subset of V. A subset B of S is a *block* of S, if the subgraph of $C(V)$ spanned by B is a connected component of the subgraph of $C(V)$ spanned by S. A block B is said to be *even* (*odd*) if $|B|$ is even (odd).

Define $\nu_V(S)$ (briefly: $\nu(S)$) to be the number of odd blocks of S $(\nu(\emptyset)=0)$. Clearly $0\leqq \nu(S)\leqq |S|$, and $\nu(S)\equiv|S| \pmod 2$. If $\nu(S)=|S|$ then we say that S is *separated.*

DEFINITION 3.1. $\mathscr{C}(V,2m)=\{T:(\exists S\subset V)[T\subset S\ \&\ |S|=2m\ \&\ \nu(S)=0]\}$. Clearly

$$\max \mathscr{C}(V,2m)=\{S\subset V:|S|=2m \text{ and } \nu(S)=0\}.$$

THEOREM 3.2. *If $V=\{1,\cdots,v\}$, $v>d=2m\geqq 2$, then $\mathscr{C}(V,2m)$ is isomorphic to $\mathscr{B}(C(v,d))$ under the correspondence $i\to x_i$.*

PROOF. It sufficies to show that $\max \mathscr{C}(V,2m)$ is isomorphic to $\max \mathscr{B}(C(v,2m))$. Suppose $S\subset V$. If $S\in \max \mathscr{C}(V,2m)$, then S clearly satisfies (∗), hence $\{x_i:i\in S\}\in \max \mathscr{B}(C(v,d))$. Conversely, if $\{x_i:i\in S\}\in \max \mathscr{B}(C(v,d))$, then $S\subset V$, $|S|=2m$ and all blocks of S which contain neither 1 nor v are even, by (∗). But S has at most one block which contains 1 or v or both, therefore $\nu(S)\leqq 1$. Since $\nu(S)\equiv|S|\equiv 0 \pmod 2$, we conclude that $\nu(S)=0$, $S\in \max \mathscr{C}(V,2m)$. □

The isomorphism type of $\mathscr{C}(V,2m)$ clearly depends only on $|V|$ and m, not on the particular choice of V. Therefore we shall usually assume that $V=\{1,\cdots,v\}$. Every automorphism of the graph $C(V)$ induces an automorphism of $\mathscr{C}(V,2m)$. The group of automorphisms of $C(V)$ is a dihedral group of order $2v$ (v rotations and v reflections). We shall see later that if $v\geqq 2m+3$, then the only automorphisms of $\mathscr{C}(V,2m)$ are those induced by automorphisms of $C(V)$.

$C(v,2m)$ can be realized as the convex hull of v evenly spaced points on the so called trigonometric moment curve (see [6, p. 67, excercise 4.8.23]). In this realization all the combinatorial automorphisms of $C(v, 2m)$ are induced by geometric symmetries (isometries).

From this point on we leave behind the polytopes $C(v, d)$ and deal mostly with the abstract complexes $\mathscr{C}(V, 2m)$ and their quotients.

THEOREM 3.3 (Shephard's condition [15]). *Suppose* $|V| = v > 2m \geqq 2$, *and* $S \subset V$. *Then* S *is a face of* $\mathscr{C}(V, 2m)$ *iff* $|S| + \nu(S) \leqq 2m$.

PROOF. (The proof given here is simpler than Shephard's original proof.) Suppose $S \subset V$. It is easily checked that if $i \in V \setminus S$, then $\nu(S \cup \{i\}) = \nu(S) \pm 1$. Therefore $|S| + \nu(S) + 2 \geqq |S \cup \{i\}| + \nu(S \cup \{i\}) \geqq |S| + \nu(S)$ and therefore, if $S \subset T \subset V$, then $|T| + \nu(T) \geqq |S| + \nu(S)$.

If $T \in \max \mathscr{C}(V, 2m)$, then $|T| + \nu(T) = 2m + 0 = 2m$. Therefore, if $|S| + \nu(S) > 2m$, then S is not a subset of a member of $\max \mathscr{C}(V, 2m)$, i.e., S is not a face of $\mathscr{C}(V, 2m)$.

Now suppose that $|S| + \nu(S) \leqq 2m$. If $|S| + \nu(S) < 2m$ and $i \in V \setminus S$, then $|S \cup \{i\}| + \nu(S \cup \{i\}) \leqq 2m$. If $|S| + \nu(S) = 2m$ and $|S| < 2m$, then $\nu(S) > 0$. If $i \in V \setminus S$ and i is adjacent to (at least one) odd block of S, then it is easily checked that $\nu(S \cup \{i\}) = \nu(S) - 1$, hence $|S \cup \{i\}| + \nu(S \cup \{i\}) = 2m$.

Thus we can enlarge S step by step until we reach a set $T, S \subset T$, with $|T| = 2m$ and $|T| + \nu(T) = 2m$, hence $\nu(T) = 0$, $T \in \max \mathscr{C}(V, 2m)$. Therefore S is a face of $\mathscr{C}(v, 2m)$. □

THEOREM 3.4. *Suppose* $|V| = v > 2m \geqq 2$. *The missing faces of* $\mathscr{C}(V, 2m)$ *are precisely the separated* $(m+1)$-*subsets of* V.

PROOF. If S is a separated $(m+1)$-subset of V, then $|S| + \nu(S) = 2m + 2 > 2m$, hence S is not a face of $\mathscr{C}(V, 2m)$; but every proper subset of S is a face of $\mathscr{C}(V, 2m)$, by Shephard's condition. Hence S is a missing face.

If $T \subset V$ is a missing face of $\mathscr{C}(V, 2m)$, then $|T| + \nu(T) \geqq 2m + 2$, and $|T \setminus \{i\}| + \nu(T \setminus \{i\}) \leqq 2m$ for all $i \in T$. It follows that $|T| + \nu(T) = 2m + 2$, and $\nu(T \setminus \{i\}) = \nu(T) - 1$ for all $i \in T$. If a block B of T has length $\geqq 2$, then the removal of a suitable point i of B will increase ν, i.e., $\nu(T \setminus \{i\}) = \nu(T) + 1$. Therefore all blocks of T are singletons, i.e., $|T| = \nu(T) = m + 1$, i.e., T is a separated $(m+1)$-subset of V. □

In the next sections we will classify the quotients $\mathscr{C}(V, 2m)/J$, where $J \subset V$. The classification will enable us to determine easily, for any two given admissible triples (V, m, J), (V', m', J'), whether or not $\mathscr{C}(V, 2m)/J$ is isomorphic to $\mathscr{C}(V', 2m')/J'$.

If J is not a face of $\mathscr{C}(V,2m)$, then $\mathscr{C}(V,2m)/J=\emptyset$; if $J\in\max\mathscr{C}(V,2m)$, then $\mathscr{C}(V,2m)/J=\{\emptyset\}$. We therefore assume that J is a face of $\mathscr{C}(V,2m)$, i.e., $|J|+\nu(J)\leqq 2m$, and that $|J|<2m$.

The isomorphism type of $\mathscr{C}(V,2m)/J$ remains unchanged if we replace J by its image under an automorphism (rotation or reflection) of the circuit $C(V)$. We may therefore assume, whenever convenient, that $V=\{1,\cdots,v\}$, $1\notin J$, and $v\in J$, unless $J=\emptyset$. Under this convention, every block of J in V is a sequence of consecutive numbers.

Now we will show that every quotient $\mathscr{C}(V,2m)/J$ is isomorphic to another quotient $\mathscr{C}(V',2m')/J'$, where J' is a separated subset of V'.

Theorem 3.5. *Suppose* $V=\{1,\cdots,v\}$, $|V|=v>2m>j$, $J\subset V$, $|J|=j$, $j+\nu(J)\leqq 2m$, $1\notin J$. *Let J' be the subset of J which contains the smallest element of each odd block of J,* $|J'|=\nu(J)$. *Let* $V'=V\setminus(J\setminus J')=(V\setminus J)\cup J'$, *and let* $2m'=2m-|J\setminus J'|=2m-(j-\nu(J))$ *(notice that m' is an integer). Then* $\mathscr{C}(V,2m)/J=\mathscr{C}(V',2m')/J'$.

Lemma 3.6. *Suppose* $V=\{1,\cdots,v\}$, $|V|=v>2m$. *If $I\subset V$ consists of two consecutive numbers, then* $\mathscr{C}(V,2m)/I=\mathscr{C}(V\setminus I,2(m-1))$.

Proof. Because of the cyclical symmetry of $\mathscr{C}(V,2m)$, it suffices to consider the case where $I=\{v-1,v\}$. This case is disposed of by applying twice the remark that follows (*) at the beginning of this section. □

Proof of Theorem 3.5. J can be represented as a disjoint union $J=J_1\cup\cdots\cup J_t\cup J'$, where $t=\frac{1}{2}(j-\nu(J))$, and each J_i $(1\leqq i\leqq t)$ consists of two consecutive numbers. Therefore

$$\mathscr{C}(V,2m)/J=\mathscr{C}(V,2m)/(J_1\cup\cdots\cup J_t\cup J')=(\cdots(\mathscr{C}(V,2m)/J_1)/J_2)/\cdots/J_t)/J'.$$

By a repeated application of Lemma 3.6 we obtain:

$$\mathscr{C}(V,2m)/J=\mathscr{C}\left(V\setminus\bigcup_{i=1}^{t}J_i,2(m-t)\right)\Big/J'=\mathscr{C}(V',2m')/J'. \qquad \square$$

A particular case of Theorem 3.5 is the following:

Corollary 3.7. *Using the notation of Theorem* 3.5, *if* $\nu_V(J)=0$, *then* $j=|J|$ *is even, and* $\mathscr{C}(V,2m)/J$ *is isomorphic to* $\mathscr{C}(V\setminus J,2m-j)$.

Theorem 3.5 allows us to restrict our attention to the case where J is a separated subset of V, and $|J|=\nu(J)\leqq m$.

THEOREM 3.8. *If* $|V| = v \geqq 2m + 3 \geqq 5$, *then the group of automorphisms of* $\mathscr{C}(V, 2m)$ *is a dihedral group of order* $2v$, *i.e., the only automorphisms of* $\mathscr{C}(V, 2m)$ *are those induced by automorphisms of the graph* $C(V)$.

PROOF. We will show that the edges of $C(V)$ can be defined in terms of $\mathscr{C}(V, 2m)$, provided $v \geqq 2m + 3$. The missing faces of $\mathscr{C}(V, 2m)$ are clearly defined in terms of $\mathscr{C}(V, 2m)$ (see Definition 2.1).

By Theorem 3.4 the missing faces of $\mathscr{C}(V, 2m)$ are exactly the separated $(m + 1)$-subsets of V. Therefore, if $a, b \in V$ are adjacent in $C(V)$, then no missing face of $\mathscr{C}(V, 2m)$ contains both a and b. But it is easy to see that if $a, b \in V$ are not adjacent in $C(V)$, then there exists a separated $(m + 1)$-subset of V which contains both a and b. (At this point of the proof we need the assumption $v \geqq 2m + 3$.) Hence $\{a, b\}$ is an edge of $C(V)$ iff no missing face of $\mathscr{C}(V, 2m)$ contains both a and b.

Therefore, every automorphism of $\mathscr{C}(V, 2m)$ preserves edges of $C(V)$, or, in other words, is induced by an automorphism of $C(V)$. □

In Sections 4 and 5 we will see that if $\mathscr{P}_i = \mathscr{C}(V_i, 2m_i)/J_i$ $(i = 1, 2)$, $|V_1| \geqq 2m_1 + 3 \geqq 5$, $|J_1| < m_1$ and J_i is separated in V_i for $i = 1, 2$, then $\mathscr{P}_1$ is isomorphic to $\mathscr{P}_2$ iff $|V_1| = |V_2|$, $m_1 = m_2$, $|J_1| = |J_2|$ and J_2 is the image of J_1 under an isomorphism between $C(V_1)$ and $C(V_2)$.

In Section 5 we will give a complete description of the quotients $\mathscr{C}(V, 2m)/J$, where J is a separated subset of V and $|V| \leqq 2m + 2$ or $|J| = m$.

4. The structure of $\mathscr{C}(V, 2m)/J$ $(v \geqq 2m + 3,\ j < m)$

In the present section we study the structure of $\mathrm{mf}(\mathscr{C}(V, 2m)/J)$, the set of missing faces of $\mathscr{C}(V, 2m)/J$. We assume, unless otherwise specified, that

$$(4.1)\qquad \begin{aligned} &V = \{1, 2, \cdots, v\}, \qquad |J| = j \leqq m, \qquad v \geqq 2m + 1 \geqq 3, \\ &\qquad\qquad J \text{ is a separated subset of } V. \end{aligned}$$

However, the main goal of this section is Theorem 4.9, which states that V, m and J can be essentially uniquely reconstructed from the quotient complex $\mathscr{C}(V, 2m)/J$, provided we know in advance that $v \geqq 2m + 3$ and $j < m$. Later, in Theorem 5.8, we will strengthen this result. We will show that the truth of the statement "$v \geqq 2m + 3$ and $j < m$" need not be known in advance, but can be decided by inspecting the quotient $\mathscr{C}(V, 2m)/J$. We will also describe what happens when $v \leqq 2m + 2$ or $j = m$.

The elements of V and J will sometimes be referred to as *vertices*, having in mind the graph $C(V)$, and by saying that two vertices of V are *adjacent* we mean that they are adjacent in the graph $C(V)$.

In order to study the missing faces of $\mathscr{C}(V, 2m)/J$ we define the concept of a *chain*. This concept will play a central role in the entire work.

DEFINITION 4.2. Let $a, b \in V \setminus J$. We say that $a \sim b$ if there is a sequence $a = a_1, a_2, \cdots, a_t = b$ $(t \geqq 1)$ such that for each $1 \leqq i < t$ there is a vertex in J adjacent to both a_i and a_{i+1}. The relation $\sim$ is easily seen to be an equivalence relation in $V \setminus J$, and the equivalence classes will be called *chains*. We will refer to these equivalence classes as "chains of $V \setminus J$", although, strictly speaking, they are defined in terms of both sets V and J. The *length* of a chain is the number of vertices in the chain. A vertex of J that is adjacent to a vertex in a chain (and hence to two different vertices, since J is separated in V) is said to be *covered* by the chain. Therefore the length of a chain exceeds by one the number of vertices of J covered by the chain. (The only possible exception, when v is even and J contains every second vertex of V, is excluded by condition (4.1).) Two different chains R_1, R_2, are *adjacent* if there are vertices $a \in R_1$, $b \in R_2$ adjacent in V. A set of chains is *separated* if no two chains in the set are adjacent.

In Fig. 1 the chains of length > 1 are marked. The sequence of lengths of successive chains, starting at the chain of length 3 and moving clockwise, is 3, 1, 2, 1, 1, 2.

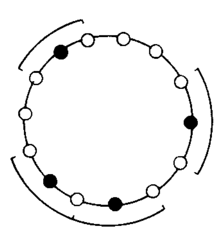

Fig. 1. The vertices in J are black, while those in $V \setminus J$ are white. The chains of length > 1 are marked.

LEMMA 4.3. *The number of chains in* $V \setminus J$ *is* $v - 2j$.

PROOF. Assume there are t chains in $V \setminus J$, and let $r_1, r_2, \cdots, r_t$ be their lengths. Then

$$v - j = \sum_{i=1}^{t} r_i \qquad \text{and} \qquad j = \sum_{i=1}^{t} (r_i - 1) = \sum_{i=1}^{t} r_i - t,$$

hence

$$t = \sum_{i=1}^{t} r_i - j = v - 2j. \qquad \square$$

For the purpose of the next theorem it will be convenient to extend the notion of a missing face as follows (see Definition 2.1):

DEFINITION 4.4. *Let* $\mathcal{D}$ *be an abstract simplicial complex, and let* V *be a set which includes* vert $\mathcal{D}$. A subset S of V is a *missing face of* $\mathcal{D}$ *relative to* V if $S \notin \mathcal{D}$, but all proper subsets of S belong to $\mathcal{D}$. Thus S is a missing face of $\mathcal{D}$ relative to V iff either $|S| \geqq 2$ and S is a missing face of $\mathcal{D}$ in the usual sense, or $|S| = 1$ and $S \subset V \setminus \text{vert}\, \mathcal{D}$.

THEOREM 4.5. *If* $v > 2m + 1$ *and* $S \subset V \setminus J$, *then* S *is a missing face of* $\mathcal{C}(V, 2m)/J$ *relative to* $V \setminus J$ *iff* S *is a union of* $m - j + 1$ *separated chains of* $V \setminus J$.

PROOF. Suppose $v > 2m + 1$ and $S \subset V \setminus J$. By Definitions 2.1 and 4.4, S is a missing face of $\mathcal{C}(V, 2m)/J$ relative to $V \setminus J$ iff $S \cup J$ is not a face of $\mathcal{C}(V, 2m)$, but for every $s \in S$ $(S \setminus \{s\}) \cup J$ is a face of $\mathcal{C}(V, 2m)$. Hence it follows from Shephard's condition (Theorem 3.3) that S is a missing face of $\mathcal{C}(V, 2m)/J$ relative to $V \setminus J$ iff

$$(*) \qquad \begin{aligned} &|S \cup J| + \nu(S \cup J) > 2m \quad \text{and} \\ &|(S \setminus \{s\}) \cup J| + \nu((S \setminus \{s\}) \cup J) \leqq 2m \quad \text{for every } s \in S. \end{aligned}$$

Since both left sides in (*) are even numbers and their difference does not exceed 2, (*) is equivalent to

$$(**) \qquad \begin{aligned} &|S \cup J| + \nu(S \cup J) = 2m + 2 \quad \text{and} \\ &|(S \setminus \{s\}) \cup J| + \nu((S \setminus \{s\}) \cup J) = 2m \quad \text{for every } s \in S. \end{aligned}$$

This is equivalent to

$$(***) \qquad \begin{aligned} &|S \cup J| + \nu(S \cup J) = 2m + 2 \quad \text{and} \\ &\nu((S \setminus \{s\}) \cup J) = \nu(S \cup J) - 1 \quad \text{for every } s \in S. \end{aligned}$$

It follows that if S is a missing face of $\mathcal{C}(V, 2m)/J$ relative to $V \setminus J$, then $S \cup J$ does not contain any even block (every even block of $S \cup J$ contains at least one element s of S, and the removal of s increases the number of odd blocks), in every block B of $S \cup J$ all the evenly-placed (i.e., the second, fourth, etc.) elements belong to J and, since J is separated in V, all the oddly-placed (i.e., the first, third, etc.) elements of B belong to S. (If an element $s \in S$ were evenly-placed in B, its removal would increase the number of odd blocks of

$S \cup J$.) Note that the restriction $v > 2m + 1$ together with (***) excludes the possibility that $S \cup J = V$.

We conclude that if S is a missing face of $\mathscr{C}(C, 2m)/J$ relative to $V \setminus J$, then S does not contain any two adjacent vertices of V, and S is a union of complete chains of $V \setminus J$, no two of which are adjacent.

On the other hand, if $S \subset V \setminus J$ is the union of a separated set of chains of $V \setminus J$ and $|S \cup J| + \nu(S \cup J) = 2m + 2$, then it is easily seen that $\nu((S \setminus \{s\}) \cup J) = \nu(S \cup J) - 1$ for every $s \in S$, and therefore S is a missing face of $\mathscr{C}(V, 2m)/J$ relative to $V \setminus J$.

Now let S be the union of t separated chains of $V \setminus J$, and let $r_1, r_2, \cdots, r_t$ be the lengths of those chains. Clearly $|S \cup J| = |S| + |J| = j + \Sigma_{i=1}^t r_i$. The number $\nu(S \cup J)$ of odd blocks in $S \cup J$ equals the number t of chains in S, plus the number x of elements of J that are not covered by any chain in S. Clearly $x = j - \Sigma_{i=1}^t (r_i - 1) = j - \Sigma_{i=1}^t r_i + t$. Therefore

$$|S \cup J| + \nu(S \cup J) = \left(j + \sum_{i=1}^{t} r_i\right) + t + \left(j - \sum_{i=1}^{t} r_i + t\right) = 2j + 2t.$$

From the preceding arguments it follows that S is a missing face of $\mathscr{C}(V, 2m)/J$ relative to $V \setminus J$ iff $2j + 2t = 2m + 2$, i.e., $t = m - j + 1$. □

LEMMA 4.6. *If $j < m$ then* vert $\mathscr{C}(V, 2m)/J = V \setminus J$.

PROOF. Clearly vert $\mathscr{C}(V, 2m)/J \subset V \setminus J$. If $x \in V \setminus J$, then $\{x\} \cup J$ is a face of $\mathscr{C}(V, 2m)$, since $|\{x\} \cup J| \leqq m$. Therefore $\{x\}$ is a face of $\mathscr{C}(V, 2m)/J$, i.e., $x \in$ vert $\mathscr{C}(V, 2m)/J$. □

Using Lemma 4.6, we obtain a simplified version of Theorem 4.5 for the case $j < m$:

THEOREM 4.7. *If $v > 2m + 1$ and $j < m$, then $S \subset V \setminus J$ is a missing face of $\mathscr{C}(V, 2m)/J$ iff S is a union of $m - j + 1$ separated chains of $V \setminus J$.*

The last theorem enables us to determine the number of missing faces of $\mathscr{C}(V, 2m)/J$, for $v > 2m + 1$ and $j < m$, as follows:

COROLLARY 4.8. *If $v > 2m + 1$ and $j < m$, then*

$$|\operatorname{mf}(\mathscr{C}(V, 2m)/J)| = \frac{v - 2j}{v - m - j - 1}\binom{v - m - j - 1}{m - j + 1}.$$

PROOF. By Lemma 4.3, the number of the chains in $V \setminus J$ is $v - 2j$. Consider those $v - 2j$ cyclically ordered chains as a cyclically ordered set V' of $v - 2j$

elements. By Theorem 4.7, a subset S' of V' induces a missing face of $\mathscr{C}(V,2m)/J$ iff $|S'| = m - j + 1$ and S' is separated in V'.

The number of separated k-subsets of a cyclically ordered n-set is

$$\left[\frac{n}{n-k}\right]\binom{n-k}{k}$$

(see [13, p. 198]). By substituting $k = m - j + 1$, $n = v - 2j$ we obtain the required result. □

We now arrive at the main goal of the present section. The question under consideration is the following: To what extent can V, J and m be reconstructed from $\mathscr{C}(V,2m)/J$ (assuming, as usual, that J is separated in V)?

THEOREM 4.9. *If $v \geqq 2m + 3$ and $j < m$, then V, J and m are essentially determined by the quotient complex $\mathscr{C}(V,2m)/J$, in the following sense:*

If $\mathscr{C}(V_1,2m_1)/J_1 \approx \mathscr{C}(V_2,2m_2)/J_2$, where $|V_i| = v_i \geqq 2m_i + 3$, $0 \leqq |J_i| = j_i < m_i$, and J_i is a separated subset of V_i for $i = 1,2$, then $m_1 = m_2$, $v_1 = v_2$, $j_1 = j_2$, and there is an isomorphism $\varphi : C(V_1) \to C(V_2)$ such that $J_2 = \varphi(J_1)$. (If $V_1 = V_2$, this means that J_2 is the image of J_1 under a rotation or reflection of $C(V_1)$.)

PROOF. Let $\mathscr{K} = \mathscr{C}(V_1,2m_1)/J_1$. By Lemma 4.6, vert $\mathscr{K} = V_1 \backslash J_1$. We will show that the chains induced by J_1 in $V_1 \backslash J_1$, as well as the cyclic order of those chains in $C(V_1)$, are determined by the structure of $\mathscr{K}$, i.e., by properties of $\mathscr{K}$ which are preserved under isomorphism.

First note that by Theorem 4.7, every missing face of $\mathscr{K}$ is the union of $m_1 - j_1 + 1$ separated chains in $V_1 \backslash J_1$. $m_1 - j_1 + 1 \geqq 2$, since $j_1 < m_1$. The number of chains in $V_1 \backslash J_1$ is $v_1 - 2j_1$, by Lemma 4.3. Since $v_1 \geqq 2m_1 + 3$, we have $v_1 - 2j_1 \geqq 2(m_1 - j_1 + 1) + 1$. Therefore, for every two chains R_1, R_2 in $V_1 \backslash J_1$ there is a missing face of $\mathscr{K}$ which contains R_1 and misses R_2. Moreover, if R_1 and R_2 are not adjacent in $C(V_1)$, then there is a missing face of $\mathscr{K}$ which contains both R_1 and R_2.

Now define an equivalence relation $\equiv$ on vert $\mathscr{K}$ as follows: $a \equiv b$ iff every missing face of $\mathscr{K}$ either contains both a and b or misses both. From the above considerations it follows that the equivalence classes of vert $\mathscr{K}$ with respect to $\equiv$ are precisely the chains of $V_1 \backslash J_1$. Moreover, two chains R_1, R_2 are adjacent in $C(V_1)$ iff no missing face of $\mathscr{K}$ contains $R_1 \cup R_2$. Thus, the chains of $V_1 \backslash J_1$ and their cyclic order in $C(V_1)$ are determined by $\mathscr{K}$.

The numbers m_1, v_1, j_1 are also determined by $\mathscr{K}$, since the maximum cardinality of faces of $\mathscr{K}$ is $2m_1 - j_1$, $|\text{vert}\, \mathscr{K}| = v_1 - j_1$, and the number of equivalence classes of vert $\mathscr{K}$ with respect to $\equiv$ is $v_1 - 2j_1$.

If $\mathscr{K}' = \mathscr{C}(V_2, 2m_2)/J_2$ is isomorphic to $\mathscr{K}$, then, by the above considerations, $m_1 = m_2$, $v_1 = v_2$, $j_1 = j_2$, and there is a $1-1$ correspondence between the chains of $V_1 \backslash J_1$ and the chains of $V_2 \backslash J_2$ which preserves size and adjacency. Such a correspondence is clearly induced by an isomorphism $\varphi : C(V_1) \rightarrow C(V_2)$ which maps J_1 onto J_2.

Note that the proof is constructive, and provides an effective procedure for producing a listing of m, V and J from a listing of $\mathscr{C}(V, 2m)/J$, when $v \geqq 2m+3$ and $j < m$. □

In view of the last theorem, it is interesting to note that the order of the vertices within each chain of $V \backslash J$ is *not* determined by $\mathscr{C}(V, 2m)/J$.

LEMMA 4.10. *Under assumption* (4.1), *every permutation* φ *of* $V \backslash J$ *which maps chains onto chains and adjacent chains onto adjacent chains induces an automorphism of* $\mathscr{C}(V, 2m)/J$.

PROOF. Let $\mathscr{K} = \mathscr{C}(V, 2m)/J$. If φ maps chains onto chains and preserves adjacency of chains, then φ maps the set of missing faces of $\mathscr{K}$ relative to $V \backslash J$ onto itself, since those missing faces are exactly all the unions of $m - j + 1$ pairwise non-adjacent chains of $V \backslash J$. Therefore φ maps $\mathscr{K}$ onto $\mathscr{K}$, since $S \in \mathscr{K}$ iff $S \subset V \backslash J$ and no subset of S is a missing face of $\mathscr{K}$ relative to $V \backslash J$. □

Combining the last lemma with a part of Theorem 4.9 we obtain:

THEOREM 4.11. *If* $v \geqq 2m+3$ *and* $j < m$, *then a permutation of* $V \backslash J$ *induces an automorphism of* $\mathscr{C}(V, 2m)/J$ *iff it maps chains onto chains and adjacent chains to adjacent chains.*

Note that for $J = \varnothing$ Theorem 4.11 reduces to Theorem 3.8.

In the sequel we regard the automorphisms of a complex (or a graph) $\mathscr{D}$ as permutations of vert $\mathscr{D}$ which map faces (or edges) of $\mathscr{D}$ onto faces (or edges, respectively) of $\mathscr{D}$. Under this convention the same permutation φ of V may be an automrophism of several different structures on V. We denote by $\mathrm{Aut}(\mathscr{D})$ the group of automorphisms of $\mathscr{D}$.

Theorem 4.11 enables us to describe the automorphisms of $\mathscr{C}(V, 2m)/J$ as follows. Define

$$\mathrm{Aut}(C(V), J) = \{\varphi \in \mathrm{Aut}\, C(V) : \varphi(J) = J\},$$

and for $\varphi \in \mathrm{Aut}(C(V), J)$, denote by $\varphi|_{V \backslash J}$ the restriction of φ to $V \backslash J$. Also let $\mathscr{S}(V \backslash J)$ be the group of all permutations of $V \backslash J$. Then:

THEOREM 4.12. *Suppose* $v \geqq 2m+3$ *and* $j < m$. *Let* $R_1, \cdots, R_{v-2j}$ *be the* *chains induced by J in $V\setminus J$. $((R_1,\cdots,R_{v-2j})$ is a partition of $V\setminus J$.) Then* $\mathrm{Aut}(\mathscr{C}(V,2m)/J)=A\cdot B$, *where*

$$A=\{\varphi\mid_{V\setminus J}:\varphi\in\mathrm{Aut}(C(V),J)\},$$

$$B=\{\psi\in\mathscr{S}(V\setminus J):\psi(R_i)=R_i \text{ for } 1\leqq i\leqq v-2j\}.$$

Therefore $|\mathrm{Aut}(\mathscr{C}(V,2m)/J)|=|\mathrm{Aut}(C(v),J)|\cdot\Pi_{i=1}^{v-2j}|R_i|!$.

PROOF. First note that $|A|=|\mathrm{Aut}(C(V),J)|$, since every automorphism of $C(V)$ is determined by its action on $V\setminus J$ (since $|V\setminus J|\geqq 3$). It is also clear that $|B|=\Pi_{i=1}^{v-2j}|R_i|!$.

If $\varphi\in\mathrm{Aut}(C(V),J)$ and $\psi\in B$, then φ and ψ map chains onto chains, and so do $\varphi\mid_{V\setminus J}$ and $\varphi_{V\setminus J}\cdot\psi$. Therefore $\varphi\mid_{V\setminus J}\cdot\psi\in\mathrm{Aut}(\mathscr{C}(V,2m)/J)$, by Theorem 4.11. This shows that $A\cdot B\subset\mathrm{Aut}(\mathscr{C}(V,2m)/J)$.

Denote by D the graph whose vertices are the chains $R_1,\cdots,R_{v-2j}$, where R_i is joined by an edge to R_j iff R_i and R_j are adjacent chains. D is a cycle of length $v-2j$. (Note that $v-2j\geqq 2m+3-2(m-1)=5$.)

Every automorphism ω of $\mathscr{C}(V,2m)/J$ induces an automorphism $\bar\omega$ of D, defined by $\bar\omega(R_i)=\omega(R_i)$, which preserves lengths of chains, i.e., $|\bar\omega(R_i)|=|R_i|$ for all i. Every such length-preserving automorphism $\bar\omega$ of D can be "built up" to an automorphism $\varphi\in\mathrm{Aut}(C(V),J)$, such that $\varphi(R_i)=\bar\omega(R_i)=\omega(R_i)$ for $R_i\in\mathrm{vert}\,D$. Let $\tilde\varphi=\varphi\mid_{V\setminus J}$, $\psi=\tilde\varphi^{-1}\cdot\omega$. Then $\tilde\varphi(R_i)=\omega(R_i)\in\mathrm{vert}\,D$ for $R_i\in\mathrm{vert}\,D$, and therefore $\psi(R_i)=R_i$ for all i. Thus $\omega=\tilde\varphi\cdot\psi$, $\tilde\varphi\in A$, $\psi\in B$.

If $\omega=\varphi_1\cdot\psi_1=\varphi_2\cdot\psi_2$, where $\varphi_1,\varphi_2\in A$, $\psi_1,\psi_2\in B$, then $\omega(R_i)=\varphi_1(R_i)=\varphi_2(R_i)$ for all i. Thus $\varphi_1^{-1}\cdot\varphi_2$ is the restriction to $V\setminus J$ of an automorphism of $C(V)$ which preserves J and maps every chain onto itself. Such an automorphism must be the identity, since the number of chains, $v-2j$, is at least 3. Therefore $\varphi_1=\varphi_2$, which implies $\psi_1=\psi_2$.

We have shown that every $\omega\in\mathrm{Aut}(\mathscr{C}(V,2m)/J)$ is uniquely expressible as a product $\varphi\cdot\psi$, $\varphi\in A$, $\psi\in B$. Therefore $|\mathrm{Aut}(\mathscr{C}(V,2m)/J)|=|A|\cdot|B|$. □

We conclude this section with an explicit description of the facets of $\mathscr{C}(V,2m)/J$ $(v\geqq 2m+3,\ j<m)$ in terms of the chains of $V\setminus J$.

THEOREM 4.13. *Suppose $v\geqq 2m+3$ and $j<m$. Let $R_1,\cdots,R_{v-2j}$ be the chains induced by J in $V\setminus J$, arranged in their natural cyclic order on $C(V)$. For $S\subset V\setminus J$, define $S^*=\{i:R_i\subset S\}$. Then $S\in\max\mathscr{C}(V,2m)/J$ iff:*

(a) $|S|=2m-j$,

(b) $|S^*|=2(m-j)$,

(c) $|R_i\setminus S|=1$ *for* $i\notin S^*$,

(d) S^* *satisfies Gale's Evenness Condition with respect to the circuit* $C(\{1,\cdots,v-2j\})$.

PROOF. First note that (b) follows from (a) and (c). Indeed, if S satisfies (a) and (c) then

$$v-2m=(v-j)-(2m-j)=|V\setminus J\setminus S|=\sum_{i=1}^{v-2j}|R_i\setminus S|$$

$$=\sum_{i\notin S^*}|R_i\setminus S|=\sum_{i\notin S^*}1=v-2j-|S^*|,$$

hence $|S^*|=2(m-j)$.

Similarly (a) and (b) imply (c), as follows: from (a) and (b) we obtain

$$v-2m=(v-j)-(2m-j)=|V\setminus J\setminus S|=\sum_{i=1}^{v-2j}|R_i\setminus S|$$

$$=\sum_{i\notin S^*}|R_i\setminus S|\geqq v-2j-|S^*|=v-2j-2(m-j)=v-2m.$$

Thus we have equality throughout, and $|R_i\setminus S|=1$ for all $i\notin S^*$.

Now assume $S\in\max\mathscr{C}(V,2m)/J$. Then clearly $|S|=2m-j$. Also, S includes no missing face of $\mathscr{C}(V,2m)/J$, and is maximal with respect to this property. Since every missing face of $\mathscr{C}(V,2m)/J$ is a union of complete chains (Theorem 4.7), S misses at most one vertex in each chain R_i. This proves (c), and (b) follows.

Define $V^*=\{1,2,\cdots,v-2j\}$. Then $S^*\subset V^*$, $|S^*|=2(m-j)$, and from Theorem 4.7 it follows that S^* does not include any separated $(m-j+1)$-subset of V^*. Thus S^* contains no missing face of the cyclic complex $\mathscr{C}(V^*,2(m-j))$ (see Theorem 3.4), hence $S^*\in\max\mathscr{C}(V^*,2(m-j))$, and therefore S^* satisfies Gale's Evennes Condition with respect to the circuit $C(V^*)$ (see Definition 3.1).

Conversely, if $S\subset V\setminus J$ satisfies conditions (a)–(d), then S^* does not include any separated $(m-j+1)$-subset of V^*, and therefore S does not include $m-j+1$ separated chains of $V\setminus J$. Thus, by Theorem 4.7, $S\in\mathscr{C}(V,2m)/J$, and since $|S|=2m-j$, $S\in\max\mathscr{C}(V,2m)/J$. □

5. Direct sums and the structure of $\mathscr{C}(V,2m)/J$ ($v\leqq 2m+2$ or $j=m$)

The main purpose of this section is to show that (under the notational convention (4.1)) if $j=m$ or $v\leqq 2m+2$ then $\mathscr{C}(V,2m)/J$ is isomorphic to the boundary complex of the direct sum of (one or more) simplices (Theorems 5.6, 5.7), whereas if $j<m$ and $v\geqq 2m+3$ then $\mathscr{C}(V,2m)/J$ is irreducible with

respect to direct sums and is not isomorphic to the boundary complex of a simplex (Theorem 5.12).

DEFINITION 5.1. Let $\mathscr{D}$ and $\mathscr{E}$ be nonempty abstract simplicial complexes, $\operatorname{vert}\mathscr{D} \cap \operatorname{vert}\mathscr{E} = \emptyset$. The *direct sum* $\mathscr{D} \oplus \mathscr{E}$ is:

$$\mathscr{D} \oplus \mathscr{E} = \{D \cup E : D \in \mathscr{D}, E \in \mathscr{E}\}.$$

The direct sum of more than two simplicial complexes is defined similarly.

Capital script letters are used in this section exclusively to denote nonempty abstract simplicial complexes.

The following properties of direct sums are easily verified:

$$\operatorname{vert}(\mathscr{D} \oplus \mathscr{E}) = \operatorname{vert}\mathscr{D} \cup \operatorname{vert}\mathscr{E}.$$

$$\max(\mathscr{D} \oplus \mathscr{E}) = \{D \cup E : D \in \max\mathscr{D}, E \in \max\mathscr{E}\}.$$

(*) $$\operatorname{mf}(\mathscr{D} \oplus \mathscr{E}) = \operatorname{mf}\mathscr{D} \cup \operatorname{mf}\mathscr{E}.$$

(**) If $D \subset \operatorname{vert}\mathscr{D}$, $E \subset \operatorname{vert}\mathscr{E}$, then

$$(\mathscr{D} \oplus \mathscr{E})/(D \cup E) = (\mathscr{D}/D) \oplus (\mathscr{E}/E).$$

In particular, if $E \in \max\mathscr{E}$ then

$$(\mathscr{D} \oplus \mathscr{E})/E = \mathscr{D}.$$

Also,

(***) $$\mathscr{D} = \{S \in \mathscr{D} \oplus \mathscr{E} : S \subset \operatorname{vert}\mathscr{D}\}.$$

Thus we see that $\mathscr{D}$ is both a subcomplex and a quotient of $\mathscr{D} \oplus \mathscr{E}$.

We say that $\mathscr{C}$ is a *factor* of $\mathscr{E}$ if $\mathscr{E} = \mathscr{C} \oplus \mathscr{D}$ for some $\mathscr{D}$. A factor of $\mathscr{E}$ is determfined by its set of vertices (see (***)). If $\emptyset \neq C \subset \operatorname{vert}\mathscr{E}$ and $\mathscr{C} = \{S \in \mathscr{E} : S \subset C\}$ is a factor of $\mathscr{E}$, then every missing face M of $\mathscr{E}$ is either included in C or disjoint from C, because of (*). Moreover, in that case

$$\operatorname{mf}\mathscr{C} = \{M \subset C : M \in \operatorname{mf}\mathscr{E}\}.$$

We say that $\mathscr{E}$ is *irreducible* if $\mathscr{E} \neq \{\emptyset\}$, and $\mathscr{E}$ has no factors other than $\mathscr{E}$ itself and $\{\emptyset\}$.

DEFINITION 5.2. Suppose $\mathscr{E} \neq \{\emptyset\}$. Define an equivalence relation $\equiv$ on $\operatorname{vert}\mathscr{E}$ as follows: $x \equiv z$ iff there are vertices $x = y_0, y_1, \cdots, y_t = z$ $(t \geqq 0)$ and missing faces $U_1, \cdots, U_t$ of $\mathscr{E}$, such that $\{y_{i-1}, y_i\} \subset U_i$ for $1 \leqq i \leqq t$. Denote by $W_1, \cdots, W_s$ the equivalence classes of $\operatorname{vert}\mathscr{E}$ with respect to $\equiv$.

If $U \in \mathrm{mf}\,\mathscr{E}$, then clearly $U \subset W_i$ for some i. If v is a vertex of $\mathscr{E}$ that does not belong to any missing face, then $\{v\} = W_i$ for some i.

From the remarks preceding Definition 5.2 we see that every factor $\mathscr{C}$ of $\mathscr{E}$ has the form $\mathscr{C} = \{S \in \mathscr{E} : S \subset C\}$, where C is a union of some of the equivalence classes $W_1, \cdots, W_s$. In particular, $\mathscr{E}$ is irreducible if $s = 1$.

Now we turn our attention to direct sums of convex polytopes.

THEOREM 5.3. *Let $V_1, \cdots, V_s$ be linear subspaces of R^d, such that $R^d = V_1 \oplus \cdots \oplus V_s$. For $1 \leqq i \leqq s$, let K_i be a convex polytope in V_i which contains the origin as an interior point, relative to V_i. Let $K = \mathrm{conv}(K_1 \cup \cdots \cup K_s)$. Then $\mathscr{B}(K) = \mathscr{B}(K_1) \oplus \cdots \oplus \mathscr{B}(K_s)$.*

PROOF. If $d = 0$, then $K = K_i = \{0\}$ and $\mathscr{B}(K) = \mathscr{B}(K_i) = \{\varnothing\}$ for all i. If $d > 0$, and $\dim V_i = 0$ for some i, then the omission of K_i will affect neither K nor the direct sum $\mathscr{B}(K_1) \oplus \cdots \oplus \mathscr{B}(K_s)$. Assume therefore that $\dim V_i > 0$ for all i.

It is easily checked that K is a d-polytope, $0 \in \mathrm{int}\, K$ and $\mathrm{vert}\, K$ is the disjoint union of $\mathrm{vert}\, K_i$, for $1 \leqq i \leqq s$.

We must show that $\mathscr{B}(K)$ consists exactly of all unions $F_1 \cup \cdots \cup F_s$, where $F_i \in \mathscr{B}(K_i)$ for $1 \leqq i \leqq s$. If $F \in \mathscr{B}(K)$, then there is a linear functional $f : R^d \to R$ such that $f(x) = 1$ for $x \in F$, $f(x) < 1$ for $x \in \mathrm{vert}\, K \setminus F$. For $1 \leqq i \leqq s$, let $F_i = F \cap \mathrm{vert}\, K_i$. Then $F_i \in \mathscr{B}(K_i)$, and $F = F_1 \cup \cdots \cup F_s$.

Conversely, if $F_i \in \mathscr{B}(K_i)$ are given for $1 \leqq i \leqq s$, let $f_i : V_i \to R$ be linear functionals such that $f_i(x) = 1$ for $x \in F_i$, $f_i(x) < 1$ for $x \in \mathrm{vert}\, K_i \setminus F_i$. Let $\tilde{f}_i : R^d \to R$ be linear functionals, such that $\tilde{f}_i(x) = f(x)$ for $x \in V_i$, $\tilde{f}_i(x) = 0$ for $x \in V_j$, $j \neq i$. Define $f = \tilde{f}_1 + \cdots + \tilde{f}_s$. Then $f(x) = 1$ for $x \in F_1 \cup \cdots \cup F_s$, $f(x) < 1$ for $x \in (\mathrm{vert}\, K_1 \setminus F_1) \cup \cdots \cup (\mathrm{vert}\, K_s \setminus F_s) = \mathrm{vert}\, K \setminus (F_1 \cup \cdots \cup F_s)$. Therefore $F_1 \cup \cdots \cup F_s \in \mathscr{B}(K)$. □

Theorem 5.3 justifies the following definition:

DEFINITION 5.4. K is the (geometric) *direct sum* of the polytopes $K_1, \cdots, K_s$, if $K = \mathrm{conv}(K_1 \cup \cdots \cup K_s)$, $\bigcap_{i=1}^{s} \mathrm{relint}\, K_i \neq \varnothing$, and $\dim K = \dim K_1 + \cdots + \dim K_s$.

From the conditions of Definition 5.4 it follows that $\bigcap_{i=1}^{s} \mathrm{relint}\, K_i$ consists of a single point z, $0 \in \mathrm{relint}(K_i - z)$ for $1 \leqq i \leqq s$, $K - z = \mathrm{conv} \bigcup_{i=1}^{s} (K_i - z)$, and $\mathrm{aff}(K - z)$ is the direct sum of the linear spaces $\mathrm{aff}(K_i - z)$, $i = 1, 2, \cdots, s$.

The next result will be useful for the classification of the quotient polytopes $C(v, 2m)/J$ in the cases $v \leqq 2m + 2$ or $|J| = m$, which are exceptions to the general case treated in Section 4.

THEOREM 5.5. *A simplicial polytope K is combinatorially equivalent to a direct sum of simplices iff the missing faces of K are pairwise disjoint.*

PROOF. (a) If T is a simplex, then $\mathrm{mf}\, T = \{\mathrm{vert}\, T\}$. Suppose K is the direct sum of simplices $T_1, \cdots, T_t$. Then $\mathcal{B}(K) = \mathcal{B}(T_1) \oplus \cdots \oplus \mathcal{B}(T_t)$, and $\mathrm{mf}\, K = \mathrm{mf}\, T_1 \cup \cdots \cup \mathrm{mf}\, T_t = \{\mathrm{vert}\, T_1, \cdots, \mathrm{vert}\, T_t\}$. The sets $\mathrm{vert}\, T_1, \cdots, \mathrm{vert}\, T_t$ are clearly pairwise disjoint.

(b) Suppose $\mathrm{mf}\, K = \{U_1, \cdots, U_t\}$, and the sets $U_1, \cdots, U_t$ are pairwise disjoint. Then $\mathrm{vert}\, K = U_1 \cup \cdots \cup U_t$ (see Lemma 2.3), and K is combinatorially equivalent to the geometric direct sum of simplices $T_1, \cdots, T_t$, of dimensions $|U_1| - 1, \cdots, |U_t| - 1$, respectively (see end of Section 2). □

Now we turn back to the cyclic polytopes $C(v, 2m)$, where $v \leqq 2m + 2$.

$C(2m+1, 2m)$ is a simplex, and all its quotients are simplices. (If T is a d-simplex and $F \in \mathcal{B}(T)$, then $\mathcal{B}(T)/F$ is the boundary complex of the $(d - |F|)$-simplex $\mathrm{conv}(\mathrm{vert}\, T \setminus F)$.)

$C(2m+2, 2m)$ is the direct sum of two m-simplices T_1, T_2 (see [6, p. 98]). More precisely, if the vertices of $C(2m+2, 2m)$ are $x_1, \cdots, x_{2m+2}$ and appear in this order on the moment curve, then

$$T_1 = \mathrm{conv}\{x_{2i-1} : 1 \leqq i \leqq m+1\} \quad \text{and} \quad T_2 = \mathrm{conv}\{x_{2i} : 1 \leqq i \leqq m+1\}.$$

By Theorem 5.3, the proper faces F of $C(2m+2, 2m)$ are exactly all the sets $F_1 \cup F_2$, where $F_1 \in \mathcal{B}(T_1)$, $F_2 \in \mathcal{B}(T_2)$, and by (**) we have

$$\mathcal{B}(C(2m+2, 2m)/F) = \mathcal{B}(C(2m+2, 2m))/F = \mathcal{B}(T_1 \oplus T_2)/F$$
$$= (\mathcal{B}(T_1) \oplus \mathcal{B}(T_2))/(F_1 \cup F_2) = \mathcal{B}(T_1)/F_1 \oplus \mathcal{B}(T_2)/F_2.$$

Following [6, p. 53], we use the symbol T^d to denote a d-simplex. Writing $T^d \oplus T^e$, we assume that the relative interiors of T^d and T^e have exactly one point in common. Thus $T^d \neq T^e$, even if $d = e$, unless $d = e = 0$.

We thus obtain:

THEOREM 5.6. *The quotients of $C(2m+2, 2m)$ are, up to isomorphism, exactly the polytopes $T^\alpha \oplus T^\beta$, where $0 \leqq \alpha \leqq \beta \leqq m$.*

Note that $T^\alpha \oplus T^\beta = T^\beta$ if $\alpha = 0$. The number of distinct types of quotients of $C(2m+2, 2m)$ is $1 + 2 + \cdots + (m+1) = \frac{1}{2}(m+1)(m+2)$.

Next we return to the quotients $\mathcal{C}(V, 2m)/J$, where $m \geqq 1$, $|V| = v \geqq 2m+3$, J is a separated subset of V, and $|J| = j = m$. Here the main result is:

THEOREM 5.7. *Let V, m, J be as above, and let $V_1, \cdots, V_t$ be the chains of* *lengths* >1 *induced by* J *in* $V\setminus J$. *If* $|V_i| = 1+\alpha_i$ *for* $1 \leqq i \leqq t$, *then* $\mathscr{C}(V,2m)/J \approx \mathscr{B}(T^{\alpha_1} \oplus \cdots \oplus T^{\alpha_t})$.

PROOF. Let $\mathscr{K} = \mathscr{C}(V,2m)/J$. By Theorem 4.5, the missing faces of $\mathscr{K}$ relative to $V\setminus J$ are precisely the chains induced by J in $V\setminus J$. By the remark that follows Definition 4.4, the missing faces of $\mathscr{K}$ relative to $V\setminus J$ are the singletons $\{x\}$, where $x \in V\setminus J \setminus \operatorname{vert} \mathscr{K}$, and the "ordinary" missing faces of $\mathscr{K}$. It follows that $\operatorname{vert} \mathscr{K} = V_1 \cup \cdots \cup V_t$, and $\operatorname{mf} \mathscr{K} = \{V_1, \cdots, V_t\}$. By Theorem 5.5 and its proof, $\mathscr{K}$ is isomorphic to the boundary complex of a direct sum of simplices $T^{\alpha_1} \oplus \cdots \oplus T^{\alpha_t}$, where $\alpha_i = |V_i| - 1$ for $1 \leqq i \leqq t$. □

REMARK. The above proof holds for $v \geqq 2m+2$. The theorem is true even for $v = 2m+1$.

The next theorem will enable us to check the truth of the statement "$v \geqq 2m+3$ and $j < m$" by looking at the quotient $\mathscr{C}(V,2m)/J$. This is the strengthening of Theorem 4.9 that was promised at the beginning of Section 4. (See also Theorem 5.12 below.)

THEOREM 5.8. *Let* V, v, J, j *and* m *be as in* (4.1). *Let* $\mathscr{K} = \mathscr{C}(V,2m)/J$. *Then the missing faces of* $\mathscr{K}$ *are pairwise disjoint iff* $v \leqq 2m+2$ *or* $j = m$.

PROOF. (a) If $v \leqq 2m+2$ or $j = m$, then $\mathscr{K}$ is isomorphic to the boundary complex of a direct sum of simplices, and the missing faces of $\mathscr{K}$ are pairwise disjoint. See Theorems 5.5, 5.6, 5.7. (This includes the case where $\mathscr{K}$ has only one missing face, and is isomorphic to the boundary complex of a simplex.)

(b) Suppose $v \geqq 2m+3$ and $j < m$. Then the missing faces of $\mathscr{K}$ are exactly the unions of $m-j+1$ separated chains of $V\setminus J$, by Theorem 4.7. There are altogether $v-2j$ chains, by Lemma 4.3. Since $m-j+1>1$ and $v-2j > 2(m-j+1)$, every chain belongs to at least two missing faces, and thus the missing faces of $\mathscr{K}$ are not pairwise disjoint. □

In Theorem 4.12 we determined the automorphisms of $\mathscr{C}(V,2m)/J$ (J separated, $|J| = j$) for the case $|V| = v \geqq 2m+3$, $j < m$. The corresponding result for the case $v \leqq 2m+2$ or $j = m$ is trivial. We include it here for the sake of completeness.

THEOREM 5.9. *Suppose* K *is the direct sum of simplices* $T_1, \cdots, T_t$. *Then* $\operatorname{vert} K = \bigcup_{i=1}^{t} \operatorname{vert} T_i$, $\operatorname{mf} K = \{\operatorname{vert} T_i : 1 \leqq i \leqq t\}$. *The automorphisms of* $\mathscr{B}(K)$ *are the permutations of* $\operatorname{vert} K$ *which preserve* $\operatorname{mf} K$. *Therefore* $|\operatorname{Aut} \mathscr{B}(K)| = \prod_{j=1}^{\infty} ((j+1)!)^{h_j} \cdot h_j!$, *where* h_j *is the number of* j*-dimensional simplices among* $T_1, \cdots, T_t$.

Theorem 5.7 enables us to enumerate the types of quotients $\mathscr{C}(V,2m)/J$ where J is a separated m-subset of V. (The corresponding enumeration problem for $|J| = j < m$ is less trivial, and will be solved in a later part of this work.) Suppose $\mathscr{C}(V,2m)/J = \mathscr{B}(T^{\alpha_1} \oplus \cdots \oplus T^{\alpha_t})$. Equating dimensions on both sides we obtain:

$$\alpha_1 + \cdots + \alpha_t = 2m - |J| = m.$$

For the number of vertices we get:

$$f_0(\mathscr{C}(V,2m)/J) = (\alpha_1 + 1) + \cdots + (\alpha_t + 1) = m + t \leqq v - j = v - m.$$

Hence $t \leqq \min(m, v - 2m)$.

Conversely, it is clear that for every sequence $\alpha_1, \cdots, \alpha_t$ of positive integers such that $\alpha_1 + \cdots + \alpha_t = m$ and $1 \leqq t \leqq v - 2m$ there is a separated m-subset J of V such that $\mathscr{C}(V,2m)/J \approx \mathscr{B}(T^{\alpha_1} \oplus \cdots \oplus T^{\alpha_t})$. (If $V = \{1, 2, \cdots, v\}$, take

$$J = J_1 \cup \cdots \cup J_t, \quad \text{where } J_i = \left\{ \sum_{k=1}^{i-1} (2\alpha_k + 1) + 2\nu : 1 \leqq \nu \leqq \alpha_i \right\}. \Big)$$

Hence:

THEOREM 5.10. *The number of different combinatorial types of quotients $\mathscr{C}(V,2m)/J$, where J is a separated m-subset of V and $|V| = v \geqq 2m + 3$, equals the number of unordered partitions of m into at most $v - 2m$ positive integers. If $v \geqq 3m$, this is the number $p(m)$ of all unordered partitions of m into positive integers.*

The function $p(m)$ is well-known in combinatorics and number theory, and is asymptotic to $(4m\sqrt{3})^{-1}\exp(\pi\sqrt{2m/3})$. (See [8, sections 4.1, 4.2].)

The next result follows easily from Theorem 3.5, Theorem 5.6 and the proof of Theorem 5.7. The proof is left to the reader.

THEOREM 5.11. *Every direct sum of simplices is isomorphic to a quotient polytope of a cyclic polytope. In fact, if $\alpha_1, \cdots, \alpha_t$ are positive integers, $t \geqq 1$, the $T^{\alpha_1} \oplus \cdots \oplus T^{\alpha_t}$ is isomorphic to a quotient polytope of $C(v, 2m)$ iff*

either $v \geqq 2m + t$ *and* $m \geqq \alpha_1 + \cdots + \alpha_t$ (Theorems 5.7 and 3.5);

or $t \leqq 2$, $v = 2m + 2$ *and* $m \geqq \max(\alpha_1, \cdots, \alpha_t)$ (Theorem 5.6);

or $t = 1$, $v = 2m + 1$ *and* $2m \geqq \alpha_1$ (trivial).

In this section we saw that if $\mathscr{K} = \mathscr{C}(V,2m)/J$, where V, m, J are as in (4.1), then $\mathscr{K}$ is isomorphic to the boundary complex of a simplex or of a direct sum of simplices iff $v \leqq 2m+2$ or $j = m$. We conclude the section by observing that if $j < m$ and $v \geqq 2m+3$, then $\mathscr{K}$ is irreducible.

THEOREM 5.12. *Using the notations of Theorem* 5.8, *if* $j < m$ *and* $v \geqq 2m+3$, *then* $\mathscr{K}$ *is irreducible, and is not isomorphic to the boundary complex of a simplex.*

PROOF. (a) From the considerations in Part (b) of the proof of Theorem 5.8 it follows that vert $\mathscr{K}(= V \setminus J)$ cannot be split into two non-empty sets W_1, W_2, such that every missing face of $\mathscr{K}$ is included in W_1 or in W_2. (The union of any two nonadjacent chains of $V \setminus J$ is included in a missing face of $\mathscr{K}$. If C_1, C_2 are adjacent chains, then there is a third chain not adjacent to C_1, nor to C_2, since $v - 2j \geqq 5$.) Therefore $\mathscr{K}$ is irreducible. (See the remarks following Definition 5.2.)

(b) vert $\mathscr{K} = V \setminus J$ by Lemma 4.6. Therefore $\mathscr{K}$ is isomorphic to the boundary complex of a $(2m-j)$-polytope K with $v-j$ vertices, and $v - j \geqq 2m - j + 3$.

6. Alternative constructions of quotients of cyclic polytopes

In this section we show how to obtain the quotients of cyclic polytopes from lower-dimensional cyclic polytopes by a process that can be described as "expansion" or "vertex-splitting". The construction is due to Schönwald [14]. We outline a direct description of this construction (Theorem 6.4), and also an indirect one, using Gale-diagrams.

DEFINITION 6.1. Let $K \subset R^d$ be a k-polytope, and let $\Delta \subset R^d$ be an r-simplex, such that aff $K \cap$ aff Δ consists of a single point p, $p \in K \cap \operatorname{relint} \Delta$. Let $K' = \operatorname{conv}(K \cup \Delta)$. Then we say that K' is *obtained from* K *by* $(r+1)$-*splitting at* p.

Note that if $p \in \operatorname{relint} K$, then $K' = K \oplus \Delta$.

THEOREM 6.2. *Let* K, k, Δ, r, p *and* K' *be as in Definition* 6.1. *Assume moreover that* K *is simplical, that* $k \geqq 1$ *and that* $p \in \operatorname{vert} K$. *Then*

(1) $\dim K' = k + r$;

(2) $\operatorname{vert} K' = (\operatorname{vert} K \setminus \{p\}) \cup \operatorname{vert} \Delta$;

(3) *if* $V \subset \operatorname{vert} K \setminus \{p\}$, $W \subset \operatorname{vert} \Delta$, *then* $V \cup W \in \mathscr{B}(K')$ *iff*
either $W \neq \operatorname{vert} \Delta$ *and* $V \in \mathscr{B}(K)$
or $W = \operatorname{vert} \Delta$ *and* $V \cup \{p\} \in \mathscr{B}(K)$;

(4) K' *is a simplicial polytope*;

(5) *if* $V \subset \operatorname{vert} K \setminus \{p\}$, $W \subset \operatorname{vert} \Delta$, *then* $V \cup W \in \operatorname{mk} K'$ *iff*
either $W = \varnothing$ *and* $V \in \operatorname{mf} K$
or $W = \operatorname{vert} \Delta$ *and* $V \cup \{p\} \in \operatorname{mf} K$.

PROOF. (1) Obvious.

(2) This is a special case of (3).

(3) The proof is easy, and is left to the reader. (A proof of (3) may also be found in [14, pp. 24–26].)

(4) This follows immediately from (3), since K is assumed to be simplicial.

(5) This follows directly from (3). The detailed verification falls naturally into a number of cases. The reasoning in the different cases is similar. We shall describe only one case. Suppose $V \cup W \in \operatorname{mf} K'$, where $V \subset \operatorname{vert} K \setminus \{p\}$, $W = \operatorname{vert} \Delta$. Then $V \cup \{p\} \notin \mathscr{B}(K)$, by (3). If $w \in W$, then $(W \setminus \{w\}) \cup V \in \mathscr{B}(K')$, hence $V \in \mathscr{B}(K)$, by (3). If $v \in V$, then $W \cup (V \setminus \{v\}) \in \mathscr{B}(K')$, hence $(V \setminus \{v\}) \cup \{p\} \in \mathscr{B}(K)$, again by (3). It follows that $V \cup \{p\} \in \operatorname{mf} K$. The remaining parts of the verification are even simpler, and are left to the reader. □

The following corollary is a useful reformulation of Theorem 6.2, part (5).

COROLLARY 6.3. *Let K, Δ, K' be as above. Define a function $f: \operatorname{vert} K' \to \operatorname{vert} K$ as follows: $f(x) = x$ for $x \in \operatorname{vert} K \setminus \{p\}$, $f(x) = p$ for $x \in \operatorname{vert} \Delta$. Then $\operatorname{mf} K' = \{f^{-1}(M) : M \in \operatorname{mf} K\}$.*

Suppose K is a simplicial k-polytope, $\operatorname{vert} K = \{p_1, \cdots, p_t\}$ $(t = f_0(K))$. Let $K = K_0, K_1, \cdots, K_t$ be a sequence of polytopes, such that each K_i $(1 \leqq i \leqq t)$ is obtained by r_i-splitting of K_{i-1} at p_i $(r_i \geqq 1)$. (Note that $p_i \in \operatorname{vert} K_{i-1}$.) Then K_t is a simplicial polytope, $\dim K_t = k + \Sigma_{i=1}^{t}(r_i - 1)$, and there is a function $f: \operatorname{vert} K_t \to \operatorname{vert} K$, such that $|f^{-1}(p_i)| = r_i$ for $1 \leqq i \leqq t$, and

$$(*) \qquad \operatorname{mf} K_t = \{f^{-1}(M) : M \in \operatorname{mf} K\}.$$

Thus, the structure of K_t does not depend on the given ordering of $\operatorname{vert} K$, and the splitting at the different vertices of K can also be done simultaneously. We say that K_t is obtained from K by r_i-splitting at p_i, for $1 \leqq i \leqq t$.

Now we shall see how a quotient of a cyclic polytope can be obtained by splitting vertices of a lower dimensional cyclic polytope.

THEOREM 6.4. *Suppose $\mathscr{K}' = \mathscr{C}(V, 2m)/J$, where V and J are as in* (4.1), *$|V| = v > 2m + 1$, $|J| = j < m$. Let $R_1, \cdots, R_{v-2j}$ be the chains induced by J in $V \setminus J$, arranged in their natural cyclic order on $C(V)$. Let K be a cyclic $2(m-j)$-polytope with $v - 2j$ vertices $x_1, \cdots, x_{v-2j}$, arranged in this order on the moment curve, and let K' be obtained from K by $|R_i|$-splitting at x_i, for $1 \leqq i \leqq v - 2j$. Then $\mathscr{B}(K') \approx \mathscr{K}'$.*

PROOF. Let X_i be the set of vertices of K' that arises from the $|R_i|$-splitting of K at x_i $(1 \leqq i \leqq v - 2j)$. Then $|X_i| = |R_i|$, and $\operatorname{vert} K'$ is the disjoint union of

$X_1, \cdots, X_{v-2j}$. Let φ be a bijection of vert K' onto vert $\mathcal{K}'(= V\setminus J = R_1 \cup \cdots \cup R_{v-2j})$ which maps X_i onto R_i $(1 \leqq i \leqq v-2j)$. In order to show that φ is an isomorphism between $\mathcal{K}'$ and $\mathcal{B}(K')$, it suffices to check that φ maps mf K' onto mf $\mathcal{K}'$ (Theorem 2.4). The missing faces of $\mathcal{K}'$ are precisely all the unions of $m-j+1$ separated chains R_i (Theorem 4.7). The missing faces of K' can be described as follows. Define f : vert $K' \to$ vert K by $f(y) = x_i$ for $y \in X_i$. Then, by (*), mf $K' = \{f^{-1}(M) : M \in \text{mf } K\}$, and $f^{-1}(M) = \bigcup\{X_i : x_i \in M\}$. Since the missing faces of the cyclic $2(m-j)$-polytope K are precisely the separated $(m-j+1)$-subsets of $\{x_1, \cdots, x_{v-2j}\}$, it follows that $\varphi(\text{mf } K') = \text{mf } \mathcal{K}'$. □

REMARKS. (a) In the above proof, we have used missing faces in order to show that $\varphi(\mathcal{B}(K')) = \mathcal{K}'$. Instead, we could have used the description of the facets of $\mathcal{K}'$ in Theorem 4.13, and the description of the facets of K' that follows from Theorem 6.2, part (3).

(b) In Theorem 6.4 we assumed that $j < m$. If $j = m$, then $\mathcal{K}'$ is isomorphic to the boundary complex of a direct sum of simplices. A direct sum of t simplices $\Delta_1, \cdots, \Delta_t$ can be thought of as obtained from a t-fold point $p = p_1 = p_2 = \cdots = p_t$ by $|\text{vert } \Delta_i|$-splitting at p_i for $1 \leqq i \leqq t$. The t-fold point p is, in a sense, a "0-dimensional cyclic polytope with t vertices".

(c) The arguments used in the proof of Theorem 6.4 actually yield also the converse of that theorem, namely: If K' is obtained from a cyclic $2k$-polytope K with t vertices $x_1, \cdots, x_t$ $(t > 2k+1 \geqq 3)$ by r_i-splitting at x_i $(1 \leqq i \leqq t)$, then $\mathcal{B}(K')$ is isomorphic to $\mathcal{C}(V, 2(k+j))/J$, where $j = \Sigma_{i=1}^{t}(r_i - 1)$, $|V| = t + 2j$, and J is a suitable separated j-subset of V.

(d) If $K' = \text{conv}(K \cup \Delta)$ is obtained from K by r-splitting at a vertex p, and K'' is obtained from K' by s-splitting at a vertex q of Δ, then K'' is obtained from k by $(r+s-1)$-splitting at p. Conversely, every r-splitting of K can be obtained by a suitable succession of $r-1$ 2-splittings.

(e) It follows that the class of quotients of cyclic polytopes is closed under the operation of splitting at a vertex. It is also closed under combinatorial equivalence and under the operation of passing to a quotient. Other classes of polytopes closed under these operations are:

(1) simplices,

(2) direct sums of simplices,

(3) simplicial polytopes.

(f) Note that the operations of splitting at a vertex and direct sum commute in the following sense: If $K = P \oplus Q$, $K' = P' \oplus Q$ and P' is obtained from P by splitting at a vertex p, then K' is obtained from K by the same splitting at p.

It follows that the class of quotients of cyclic polytopes is the smallest class that includes the cyclic polytopes and the d-octahedra ($d = 0, 1, 2, \cdots$) and is closed under 2-splitting at a vertex and under combinatorial equivalence.

(g) Splitting is not a purely combinatorial operation. I.e., if K' is obtained from K by splitting at a vertex, and K'' is combinatorially equivalent to K', then K'' is not necessarily obtained from a lower-dimensional polytope by splitting.

Next we show that the operation of splitting K at a vertex is nicely reflected as a simple operation of "multiplying a vertex" in the Gale-diagram of K. It will follow (Theorem 6.9) that every simplicial d-polytope with $d+3$ vertices is a quotient of a cyclic polytope.

In the sequel we shall freely use the notation and results of [6, section 5.4] concerning Gale-diagrams.

THEOREM 6.5. *Let K be a polytope with t vertices $p_1, \cdots, p_t$. Suppose $K' = \operatorname{conv}(K \cup \Delta)$ is obtained from K by $(r+1)$-splitting at p_t. Thus*

$$\operatorname{vert} K' = \{p_1, \cdots, p_{t-1}, q_t, \cdots, q_{t+r}\}, \quad \textit{where } \{q_t, \cdots, q_{t+r}\} = \operatorname{vert} \Delta.$$

If $(\hat{p}_1, \cdots, \hat{p}_t)$ is a Gale-diagram of K, then

$$(\hat{p}_1, \cdots, \hat{p}_{t-1}, \underbrace{\hat{p}_t, \cdots, \hat{p}_t}_{r+1 \text{ times}})$$

is a Gale-diagram of K'.

LEMMA 6.6. *Let $P = (p_1, \cdots, p_t)$ be a sequence of (not necessarily distinct) points in R^d. Let $A = \operatorname{aff}\{p_1, \cdots, p_t\}$, and let $L \subset R^d$ be an r-flat, such that $A \cap L = \{p_t\}$. Let $q_t, \cdots, q_{t+r}$ be the vertices of an r-simplex Δ in L that includes p_t in its relative interior. I.e., $p_t = \beta_t q_t + \cdots + \beta_{t+r} q_{t+r}$, where $\beta_i > 0$ for $t \leqq i \leqq t+r$ and $\beta_t + \cdots + \beta_{t+r} = 1$. If $(\bar{p}_1, \cdots, \bar{p}_t)$ is a Gale-transform of the sequence $(p_1, \cdots, p_t)$, then $(\bar{p}_1, \cdots, \bar{p}_{t-1}, \beta_t \bar{p}_t, \cdots, \beta_{t+r} \bar{p}_t)$ is a Gale-transform of the sequence $P' = (p_1, \cdots, p_{t-1}, q_t, \cdots, q_{t+r})$.*

From this lemma it follows that if $(\hat{p}_1, \cdots, \hat{p}_t)$ is a Gale-diagram of P, then

$$(\hat{p}_1, \cdots, \hat{p}_{t-1}, \underbrace{\hat{p}_t, \cdots, \hat{p}_t}_{r+1 \text{ times}})$$

is a Gale-diagram of P'. This includes the assertion of Theorem 6.5 as a particular case.

PROOF OF LEMMA 6.6. Suppose $\dim A = k$. By an *affine dependence* (a.d.) of P we mean a sequence $(\alpha_1, \cdots, \alpha_t) \in R^t$ such that $\alpha_1 p_1 + \cdots + \alpha_t p_t = 0$,

$\alpha_1 + \cdots + \alpha_t = 0$. The set of all a.d.'s of P is a linear subspace of R^t of dimension $t - k - 1$.

From the assumptions of the lemma it follows that

$$\dim \operatorname{aff}\{p_1, \cdots, p_{t-1}, q_t, \cdots, q_{t+r}\} = \dim \operatorname{aff}(A \cup \Delta) = k + r,$$

and the dimension of the linear space of all a.d.'s of P' is again $t - k - 1$ $(= (t + r) - (k + r) - 1)$. It is easily checked that if $(\alpha_1, \cdots, \alpha_t)$ is an a.d. of P, then $(\alpha_1, \cdots, \alpha_{t-1}, \beta_t\alpha_t, \cdots, \beta_{t+r}\alpha_t)$ is an a.d. of P'. Therefore, if the columns of the matrix

$$D_1 = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,t-k-1} \\ & \vdots & \\ \alpha_{t,1} & & \alpha_{t,t-k-1} \end{pmatrix}$$

from a basis of the space of a.d.'s of P, then the columns of the matrix

$$D_2 = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,t-k-1} \\ \vdots & & \vdots \\ \alpha_{t-1,1} & & \alpha_{t-1,t-k-1} \\ \beta_t\alpha_{t,1} & & \beta_t\alpha_{t,t-k-1} \\ \vdots & & \vdots \\ \beta_{t+r}\alpha_{t,1} & & \beta_{t+r}\alpha_{r,t-k1} \end{pmatrix}$$

form a basis of the space of a.d.'s of P'. We can choose D_1 to be the matrix whose rows are precisely the vectors $\bar{p}_1, \cdots, \bar{p}_t$ of the given Gale-transform of P. In that case, the rows of D_2 will be $\bar{p}_1, \cdots, \bar{p}_{t-1}, \beta_t\bar{p}_t, \cdots, \beta_{t+r}\bar{p}_t$, and this sequence of vectors is the required Gale-transform of P'. □

By repeated application of Theorem 6.5 we obtain the following corollary to Theorem 6.4

Corollary 6.7. *Using the notations of Theorem 6.4, assume that $(\hat{x}_1, \cdots, \hat{x}_{v-2j})$ is a Gale-diagram of the cyclic polytope K $(= C(v - 2j, 2(m - j)))$. Then*

$$(\underbrace{\hat{x}_1, \cdots, \hat{x}_1}_{r_1 \text{ times}}, \cdots, \underbrace{\hat{x}_{v-2j}, \cdots, \hat{x}_{v-2j}}_{r_{v-2j} \text{ times}})$$

is a Gale-diagram of K'.

The analogue of Theorem 6.4 for the case $j = m$ is contained in Remark (b) above. The reformulation of that remark in terms of Gale-diagrams runs as follows:

COROLLARY 6.8. *Suppose* $K' = T_1 \oplus \cdots \oplus T_t$ *is a geometric direct sum (in the sense of Definition* 5.4) *of t simplices* T_i, *and* $\dim T_i = r_i - 1 \geqq 1$ *for* $1 \leqq i \leqq t$. *Then K' has a Gale-diagram of the form*

$$(\underbrace{\hat{p}_1, \cdots, \hat{p}_1}_{r_1 \text{ times}}, \cdots, \underbrace{\hat{p}_t, \cdots, \hat{p}_t}_{r_t \text{ times}}),$$

where $(\hat{p}_1, \cdots, \hat{p}_t)$ *is a Gale-diagram of a constant sequence*

$$\underbrace{(p, \cdots, p)}_{t \text{ times}}.$$

Note that $(\hat{p}_1, \cdots, \hat{p}_t)$ *is a Gale-diagram of a constant sequence iff* $\hat{p}_1, \cdots, \hat{p}_t$ *are unit vectors in* R^{t-1} *and* $0 \in \operatorname{int} \operatorname{conv}\{\hat{p}_1, \cdots, \hat{p}_t\}$.

We conclude this section with a theorem of F. Hering [9, theorem 2.16 and its corollary on page 140].

THEOREM 6.9. *Every simplicial d-polytope K with* $d + \beta$ *vertices,* $1 \leqq \beta \leqq 3$, *is a quotient of a cyclic polytope* $C(2m + \beta, 2m)$. *(m depends on K.)*

PROOF. The cases $\beta = 1$ and $\beta = 2$ are trivial.

Let K be a d-polytope with $d + 3$ vertices. By [6, page 109], K is combinatorially equivalent to a polytope K' with a contracted Gale-diagram G'. G' consists of an odd number $2\mu + 3$ points $\hat{p}_1, \cdots, \hat{p}_{2\mu+3}$ evenly spaced on the unit circle,

$$\hat{p}_i = \left(\cos \frac{i-1}{2\mu+3} \cdot 2\pi, \sin \frac{i-1}{2\mu+3} \cdot 2\pi\right) \quad \text{say,}$$

with positive multiplicities $r_1, \cdots, r_{2\mu+3}$ (see, e.g. [6, fig. 6.3.2]). The sequence $G'' = (\hat{p}_1, \cdots, \hat{p}_{2\mu+3})$ without multiplicities is the Gale-diagram of a cyclic polytope $C(2\mu + 3, 2\mu)$ if $\mu > 0$, or of a triple point (p, p, p) if $\mu = 0$.

If $\mu > 0$, then G' is a Gale-diagram of a polytope that is obtained from $C(2\mu + 3, 2\mu)$ by splitting at vertices (Theorem 6.5), and such a polytope is a quotient of a cyclic polytope $C(2m + 3, 2m)$ (see Remark (c) above). An easy calculation shows that $m = d - \mu$.

If $\mu = 0$, then G' is a Gale-diagram of a direct sum of three simplices $T^{r_1-1} \oplus T^{r_2-1} \oplus T^{r_3-1}$, which is a quotient of $C(2d + 3, 2d)$, since $d = r_1 - 1 + r_2 - 1 + r_3 - 1$ (see Theorem 5.11).

In both cases we showed that K is a quotient of $C(2m + 3, 2m)$ for some m, $\frac{1}{2}d \leqq m \leqq d$. □

REMARK. For $d \geqq 3$ and $\beta \geqq 4$, only a small minority of the simplicial d-polytopes with $d+\beta$ vertices are quotients of cyclic polytopes.

As in [6], denote by $c_s(d+\beta, d)$ the number of combinatorial types of simplicial d-polytopes with $d+\beta$ vertices, and let $q(d+\beta, d)$ be the number of those which are types of quotients of cyclic polytopes. Then it can be shown quite easily that $q(d+\beta, d)$ increases much slower than $c_s(d+\beta, d)$ for fixed d ($d \geqq 3$) and increasing β, as well as for fixed β ($\beta \geqq 4$) and increasing d. E.g., $q(8,4)=6$, $q(9,4)=5$, whereas $c_s(8,4)=37$ (see [7]), and $c_s(9,4)=1142$ (see [2]).

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